Probability Analysis Applied to a Water-Supply Problem

By Luna B. Leopold

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INTRODUCTION

The literature on probability techniques applicable to problems in hydrology is abundant but scattered through scientific journals of both hydrology and statistics. Important administrative and judicial decisions presently face water-compact commissions, courts, and water-planning committees. These and other groups might find useful a brief and simplified discussion of how statistical techniques can aid in analysing problems of water supply. The interest expressed in this subject by various parties to the litigation concerning the Colorado River prompts this publication of material, which was presented in August 1958 before the Special Master of the Supreme Court hearing the proceedings of Arizona v. California et al. The examples presented here are the same as those used in testimony before the Special Master, but there are included the basic computations, which were too detailed to present in the actual testimony.

The specific example, which was analyzed in that testimony, was a 61-year series of annual discharge values of the Colorado River at Lees Ferry, 1896 to 1956, inclusive. However, the methodology presented herein is generally applicable to many other streamflow records; and the specific data discussed should be viewed as exemplifying the types of information, which can be obtained from any streamflow record.

The series of data used are the 61 years of reconstructed record of annual discharge values representing the so-called virgin flow of the Colorado River at Lees Ferry.[1] This particular series was used by witnesses from Arizona and California in the Colorado River litigation (California exhibit 2201A). The series was compiled from three sources. For the period 1896 to 1947 the annual discharges were derived from the U. S. Bureau of Reclamation "Report on Colorado River Storage Project and Participating Projects," dated December 1950. The series from 1948 through 1951 was obtained from the Bureau of Reclamation "Memorandum Supplement to Report on Water Supply of the Lower Colorado River Basin, Project Planning Report, November 1952," dated November 1953. The annual discharges for 1952 through 1956 were derived by the same methods used th the Bureau of Reclamation. The entire series from California exhibit 2201A is listed in columns 1 and 2 of table 1. The remaining columns are explained on page 4.

It is generally understood that the mean flow experienced during the period of record at a given gaging station will not necessarily be duplicated in future periods. However, it is not as well recognized that proper analysis can yield much information in addition to the mean flow for the period of record. Simple statistical techniques can be used to obtain values of the probability that any specific flow will be equaled or exceeded in the future, or that any particular value will not be reached in future periods. The objectively determined probabilities will not dictate a particular course of action of decision, but they do at least provide a framework within which decisions on water-supply problems can be made.

It should be emphasized that a statement of probability is not a forecast. Extensive studies of the variation of hydrologic phenomena clearly indicate that values of any hydrologic factor tend to vary with time, but these variations are not sufficiently regular to be deemed cyclic. For forecasting future hydrologic events there must be repetitive cyclical phenomena; cyclic phenomena imply that at some time in the future the experience of the past will tend to be duplicated. Repetitive cycles are, for all practical purposes, absent in hydrologic data. Therefore, the past record should be used as an indication only of the probability that certain events will occur in the future, not as a forecast.

[1]The division between the upper basin and lower basin of the Colorado River, as defined in the Colorado River compact of 1922, is a point on the Colorado River 1 mile downstream from the Paria River. This point is called Lee Ferry in the compact. Lees Ferry is the name of a nearby place where the Geological Survey makes river measurements.

Table 1.—Reconstructed annual flows of Colorado River at Lees Ferry, Ariz.

[Flow represents discharge adjusted for upstream depletion]

(1) Water year[1]	(2) Annual flow (in thousands of acre-feet)	(3) Deviation from mean (in thousands of acre-feet)	(4) Square of deviation from mean ($n \times 10^{10}$)
1896	10,089	-5,091	2,590
97	18,009	+2,829	801
98	13,815	-1,365	186
99	15,874	+694	48
1900	13,228	-1,952	381
01	13,582	-1,598	255
02	9,393	-5,787	3,340
03	14,807	-373	13
04	15,645	+465	22
1905	16,027	+847	72
06	19,121	+3,941	1,550
07	23,402	+8,222	6,840
08	12,856	-2,324	540
09	23,275	+8,095	6,550
1910	14,248	-932	87
11	16,028	+848	72
12	20,520	+5,340	2,860
13	14,473	-707	50
14	21,222	+6,042	3,650
1915	14,027	-1,153	133
16	19,201	+4,021	1,615
17	24,037	+8,857	7,840
18	15,364	+184	3
19	12,462	-2,718	735
1920	21,951	+6,771	4,590
21	23,015	+7,835	6,060
22	18,305	+3,125	974
23	18,269	+3,089	954
24	14,201	-979	94
1925	13,033	-2,147	461
26	15,853	+673	45
27	18,616	+3,436	1,180
1928	17,279	+2,099	438
1929	21,428	+6,248	3,900
1930	14,885	-295	9
31	7,769	-7,411	5,480
32	17,243	+2,063	425
33	11,356	-3,824	1,461
34	5,640	-9,540	9,100
1935	11,549	-3,631	1,318
36	13,800	-1,380	190
37	13,740	-1,440	207
38	17,545	+2,365	560
39	11,075	-4,105	1,685
1940	8,601	-6,579	4,330
41	18,148	+2,968	884
42	19,125	+3,945	1,555
43	13,103	-2,077	430
44	15,154	-26	0
1945	13,410	-1,770	314
46	10,426	-4,754	2,260
47	15,473	+293	9
48	15,613	+433	19
49	16,376	+1,196	143
1950	12,894	-2,286	522
51	11,647	-3,533	1,250
52	20,290	+5,110	2,610
53	10,670	-4,510	2,035
54	7,900	-7,280	5,300
1955	9,150	-6,030	3,640
56	10,720	-4,460	1,990
Sum	925,957	....	106,655

[1] 12 months ending September 30 of year shown.

VARIABILITY

Probability analysis is in essence an analysis of the variability of a sample. A streamflow record represents a time sample out of an indefinitely long time period. Therefore, more information can be obtained if the data from that record are treated as other sampling data.

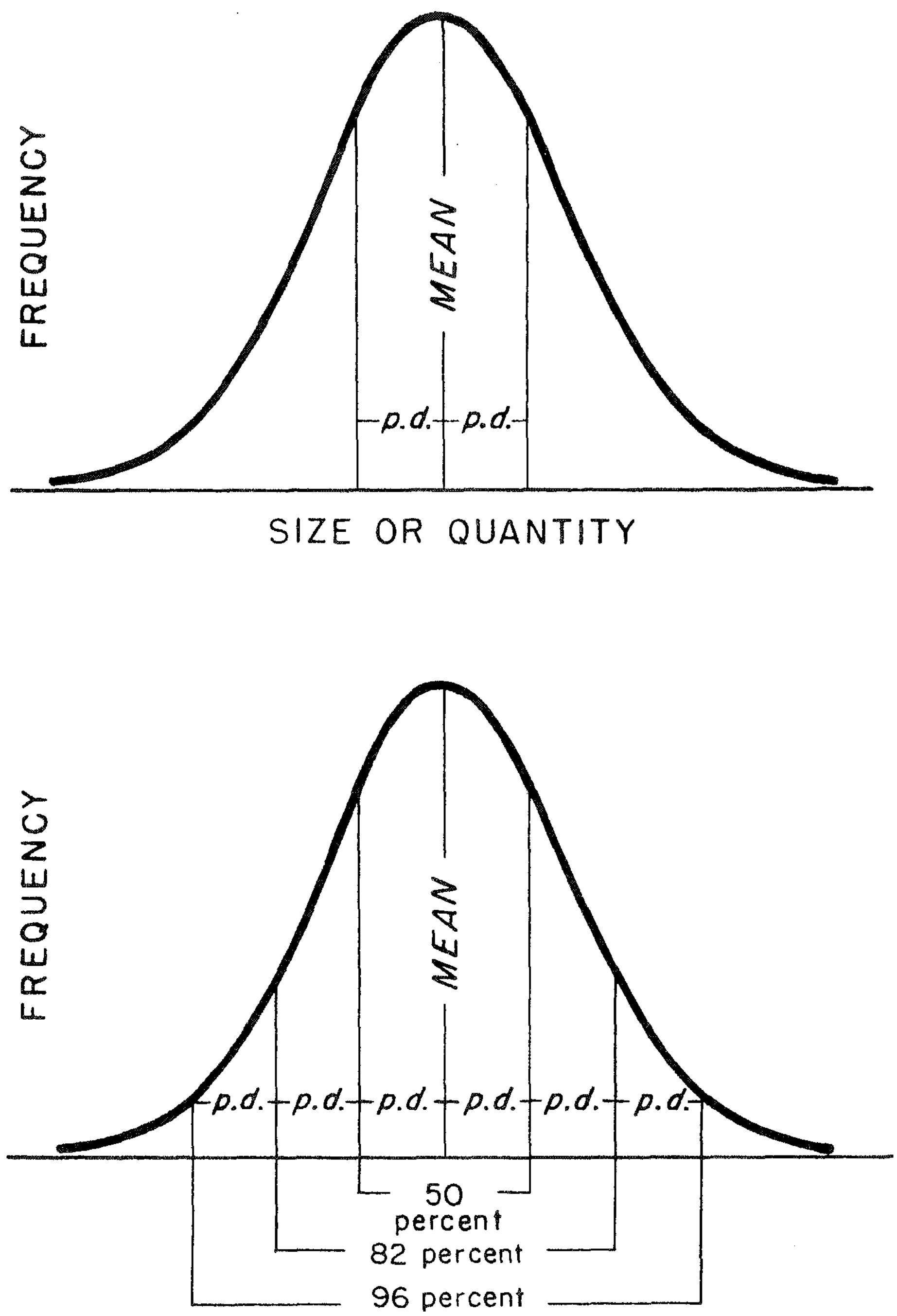
To determine the characteristics of any large population by taking a sample, the most obvious parameter indicated by the sample is the mean value. Of equal significance is the spread or dispersion of individual values about the arithmetic mean. In streamflow, for example, the annual discharge values include a few exceptionally large ones, a few small ones, and a preponderance of discharges centered around some central value. The distribution of annual discharges is shown graphically by a histogram that shows the number of years in which the discharge falls into different categories of size.

When the distribution of sizes or quantities in a population follows the so-called normal law, the histogram will present the shape of the normal distribution. A smooth curve drawn through the points on the histogram will result in a bell-shaped curve, such as is shown in figure 1. A normally distributed population is; by definition, one whose histogram can be approximated by the bell-shaped curve illustrated.

Normal distributions have certain standard characteristics. One is that the arithmetic mean of the values should be identical with the mode, which is that category having the largest number of cases. This characteristic will be present only when the bell-shaped curve is symmetrical. The symmetry of the bell-shaped curve is direct evidence that 50 percent of the cases have values higher than the mean, and 50 percent have values lower than the mean.

By the same reasoning, a characteristic of a normal distribution is that a spread of values may be defined within which 50 percent of the individual values fall. Such a spread is defined as one probable deviation on each side of the mean. This spread is is indicated on the graph of figure 1.

For the same reason a given percentage of the total cases will fall within the limits defined by two probable deviations on either side of the mean. It is a characteristic of normally distributed populations that 82 percent of the total number of cases will lie within 2 probable deviations on either side of the mean, and that 96 percent of the cases will lie within the limits of 3 probable deviations on either side of the mean.



NOTE:
p.d. = probable deviation
Arithmetic average (mean) = size class having largest number of cases (mode)

Figure 1.—The normal distribution: The frequency of occurrence of various sizes or quantities.

The bell-shaped curve of a normal distribution is asymtotic to the abscissa or horizontal coordinate; that is, the two tails of the bell-shaped curve gradually approach, but do not reach, the horizontal line of the graph.

The probable deviation may be computed for a normally distributed population in a simple manner. One probable deviation equals 0.6745 times a quantity called the standard deviation. The standard deviation[2] may be computed by the following formula:

$$\text{Standard deviation} = \sqrt{\frac{\Sigma\ x^2}{n-1}}\ ;$$

where x is the difference between the value of an individual measurement and the mean of all the measurements in a sample, and n is the number of measurements in the sample.

An example of the computation of the standard deviation is given in table 1. Column 2 in table 1 shows the annual flow in acre-feet at Lees Ferry for each water year. The arithmetic mean of column 2 is computed to be 15,180,000 acre-feet. Deviations from the mean shown in column 3 are merely the difference between the individual annual discharge values and the above-mentioned mean. Column 4 is a tabulation of the square of the deviations or the square of each value in column 3. The probable deviation is indicated as the product of the factor 0.6745 times the standard deviation, or

$$\text{Probable deviation} = 0.6745 \times 4.22 = 2.84 \text{ million acre-feet}$$

By definition, therefore, one probable deviation on either side of the mean would include 50 percent of all the values listed in column 2. That is, 50 percent of the values in column 2 lie within the limits of 18.02 and 12.34 million acre-feet. The data in column 2 will verify this statement approximately.

[2]The sums (table 1) of the annual flow, in thousands of acre-feet, and the square of deviation from the mean are 925,957 and $106{,}655 \times 10^{10}$, respectively. The mean is derived by dividing the sum of the annual flow by the total number of years from 1896–1956, thus

$$\text{Mean} = \frac{925{,}957}{61} = 15.180 \text{ million acre-feet.}$$

The computation of the standard deviation is as follows:

$$\text{Standard deviation} = \sqrt{\frac{\Sigma\ x^2}{n-1}}\ ;$$

by substituting values for symbols, the equation then simplifies to

$$\sqrt{\frac{106{,}655 \times 10^{10}}{61-1}} = \sqrt{1777 \times 10^{10}}\ ;$$

therefore,

Standard deviation = 4.22 million acre-feet.

Probability analysis is greatly simlified where the data in the sample are distributed in a normal manner.[3] That the data shown in table 1 are normally distributed will be shown, after a discussion of the use of probability paper for plotting.

PROBABILITY PLOTTING

For ease of analysis, the bell-shaped graph of a normal distribution can be plotted in a somewhat different manner by accumulating progressively the number of cases equal to, or less than, any particular value. Such a cumulative curve is shown in figure 2A. For the vertical scale, probable deviations on either side of the mean are used. Note that, as indicated above, in defining a normal distribution 50 percent of the cases are greater than the mean and, therefore, the value of 50 percent on the abscissa corresponds to 0 on the ordinate scale. One probable deviation below the mean should also correspond with 25 percent of the cases, inasmuch as the spread between the mean and one probable deviation below the mean must consist of 25 percent of the total values. Thus, the 25-percent point on the abscissa corresponds to -1 probable deviation on the ordinate; and, similarly, 75 percent on the abscissa corresponds to one probable deviation above the mean on the ordinate. It was also stated that one probable deviation on either side of the mean will include 50 percent of the total cases; on figure 2 the difference between 75 percent and 25 percent is 50 percent of the total. Also note that in figure 2A the cumulative distribution approaches but does not reach abscissa values of 0 and 100.

In order to further simplify the cumulative frequency distribution curve, the abscissa scale can be changed by spreading out the values in such a way that the S-shaped line of figure 2A becomes a straight line, as shown in figure 2B. The expanded horizontal scale should be such that the characteristics of a normal distribution are fulfilled. As an example, 82 percent of the cases should lie within two probable deviations on either side of the mean. Thus, the value of -2 on the ordinate scale will appear at an abscissa value of 9 percent, and +2 on the ordinate scale will appear opposite 91 percent on the abscissa.

The abscissa scale so constructed that a normal cumulative frequency distribution will plot as a straight line is, by definition, a probability scale. Graph paper in which the probability scale is printed as the abscissa is widely used and can be purchased at most engineering-supply stores.

To test whether a series of data is normally distributed, the values may be arranged in order of magnitude and plotted on probability paper. Table 2 lists the annual flows of the Colorado River in order of magnitude, beginning with the largest. If such data aline

[3]The annual discharges of all rivers are not normally distributed. In general, however, the annual flow of the larger streams, and those in humid regions tends toward normality.

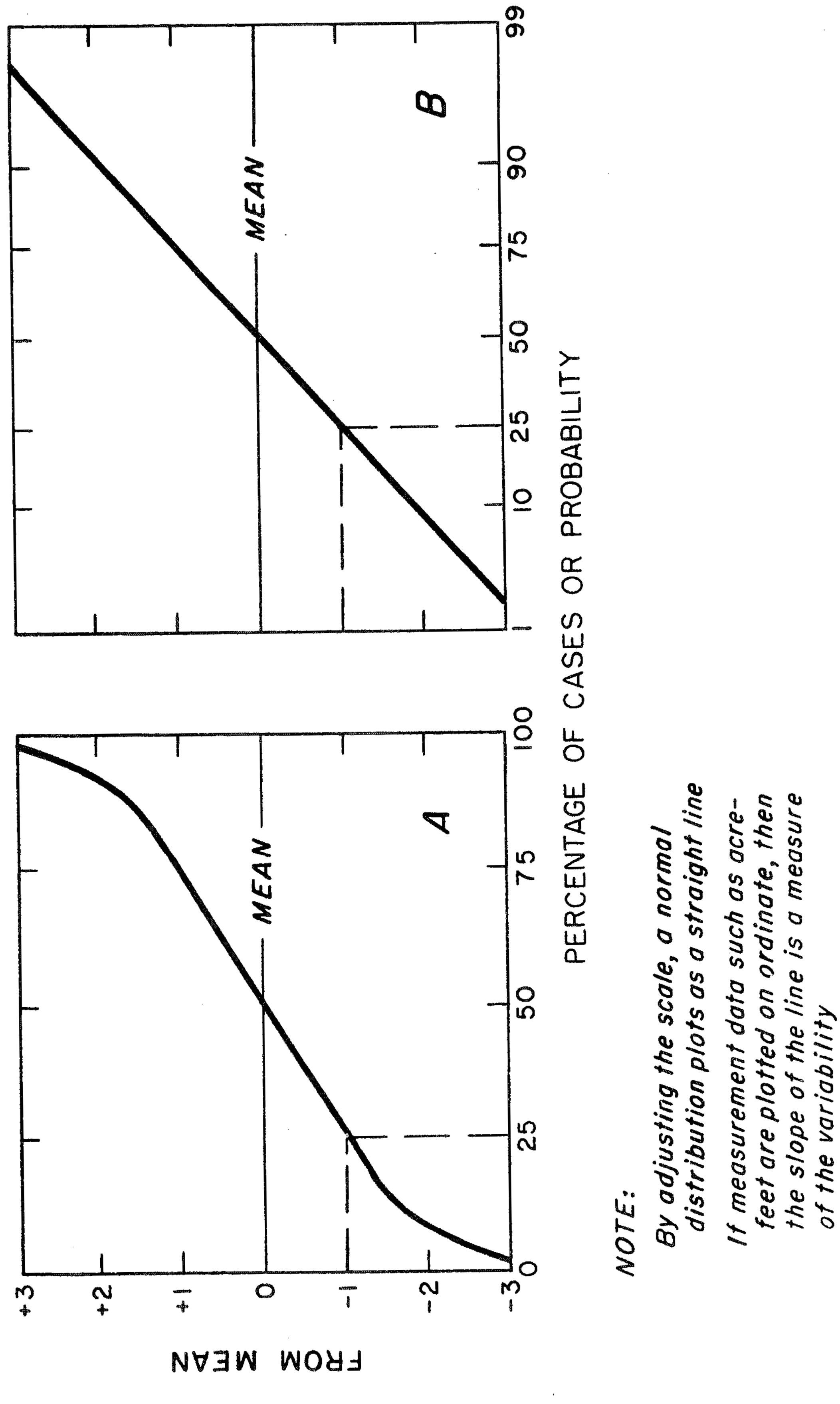


Figure 2.—The cumulative frequency distribution of a normal population.

themselves approximately in a straight line on probability paper, the values in the sample may be considered to be normally distributed. In making such a graph, the abscissa position of any individual value is obtained by using the formula

$$\text{Plotting position} = \frac{\underline{m}}{\underline{n} + 1},$$

in which m is the rank of the individual number in the array and n is the total number of cases in the sample. The plotting positions for data in table 2 are computed by this formula and shown in column 3. In table 2, 1 year of 61 years constitutes 1.6 percent of the total sample. Having arranged the 61 values in order of magnitude, as shown in column 2, the computed plotting position places each point in abscissa positions 1.6 percent apart.

The data in table 2, plotted on arithmetic probability paper, are shown in figure 3, where the 61 points aline themselves in a reasonable approximation to a straight line, as illustrated by a line drawn to conform with the points.

As indicated previously (p. 4), the probable deviation is computed to be 2.84 million acre-feet; 2.84 million acre-feet on either side of the mean (fig. 3) should include 50 percent of the total values in table 2. The graph in figure 3 confirms this computation. The ordinate value of 12.34 corresponds to an abscissa value of 25 percent and 18.02 corresponds to an abscissa value of 75 percent.

The value of one probable deviation can be obtained by reading the difference between the ordinates corresponding to 50 percent and 25 percent on the plotted graph, without going through the numerical computation shown on page 4.

Not all hydrologic data are normally distributed. Such data should be transformed—to use a statistical term—to provide a series of values that are normally distributed. In some cases, though the individual values are not normally distributed, the logarithms of those values will be normally distributed. The handling of log-normal distributions is beyond the scope of this report.

Table 2.—Computation of plotting position in probability analysis, annual flows of Colorado River at Lees Ferry

(1) Serial	(2) Annual flow in order of magnitude (thousands of acre-feet)	(3) Plotting position (probability)	(1) Serial	(2) Annual flow in order of magnitude (thousands of acre-feet)	(3) Plotting position (probability)
1	24,037	98.4	32	14,807	48.4
2	23,402	96.1	33	14,473	46.8
3	23,275	95.2	34	14,248	45.2
4	23,015	93.6	35	14,201	43.6
5	21,951	92.0	36	14,027	42.0
6	21,428	90.4	37	13,815	40.4
7	21,222	88.8	38	13,800	38.7
8	20,520	87.1	39	13,740	37.1
9	20,290	85.5	40	13,582	35.4
10	19,201	83.8	41	13,410	33.8
11	19,125	82.2	42	13,228	32.2
12	19,121	80.6	43	13,103	30.6
13	18,616	79.0	44	13,033	29.0
14	18,305	77.4	45	12,894	27.4
15	18,269	75.8	46	12,856	25.8
16	18,148	74.2	47	12,462	24.2
17	18,009	72.5	48	11,647	22.5
18	17,545	70.9	49	11,549	21.0
19	17,279	69.3	50	11,356	19.3
20	17,243	67.7	51	11,075	17.7
21	16,376	66.1	52	10,720	16.2
22	16,028	64.5	53	10,670	14.4
23	16,027	62.9	54	10,426	12.8
24	15,874	61.3	55	10,089	11.2
25	15,853	59.7	56	9,393	9.6
26	15,645	58.1	57	9,150	8.0
27	15,613	56.4	58	8,601	6.4
28	15,473	54.8	59	7,900	4.8
29	15,364	53.2	60	7.769	3.2
30	15,154	51.6	61	5,640	1.6
31	14,885	50.0			

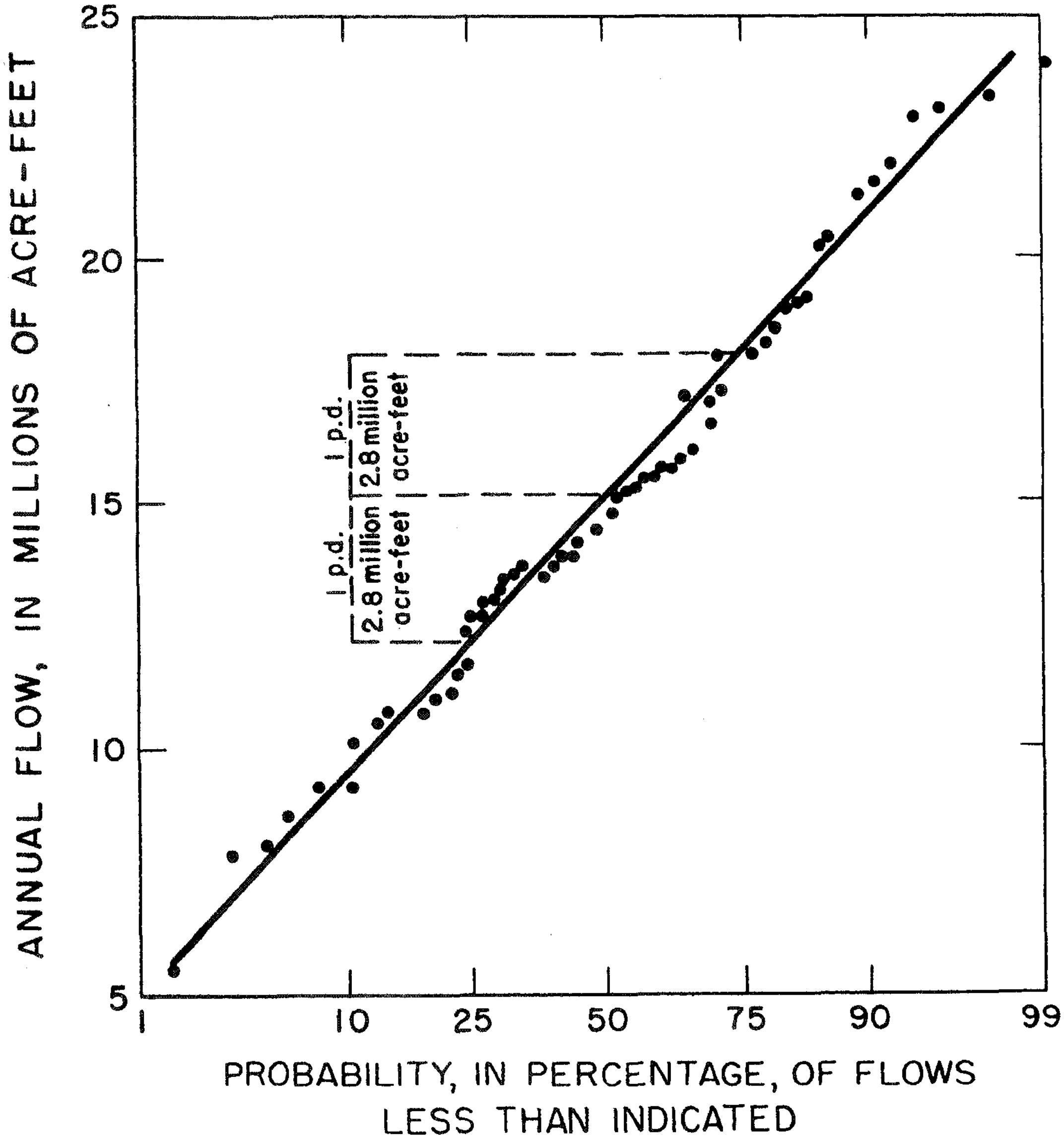


Figure 3.—Cumulative distribution curve, Colorado River at Lees Ferry, 61 years—1896–1956.

In data obtained by sampling, where the individual values in the sample are normally distributed, the means of groups of data in the sample will also be normally distributed. For example, in a streamflow record the means of 10-year periods may be computed, and the values of 10-year means may be treated as items in another sample. In a 61-year record only 6 independent 10-year means may be computed—a relatively small sample. However, the characteristics of the whole population may be approximated by the characteristics of a sample because the "standard error of the mean" of the sample is a close approximation to the standard deviation of the means of other samples from the same population. The standard error of the mean is

$$S_{\bar{x}} = \frac{\text{Standard deviation}}{\sqrt{n}},$$

where n is the number of items comprising the sample.

The standard error of the 61-year mean is computed to be

$$S_{\bar{x}} = \frac{4.22}{\sqrt{61}} = 0.54 \text{ million acre-feet.}$$

The probable error is

$$0.6745 \times 0.54 = 0.364 \text{ million acre-feet.}$$

This figure can be interpreted in two ways: (a) as a 50-percent chance that the 61-year mean of record lies within 0.364 million acre-feet of the true mean of an indefinite series of years or (b) as the probable deviation of an indefinitely large number of 61-year means.

However, this calculation is based on an assumption that the data occur in random order—a requirement not actually met in hydrologic data. That is, neither the individual annual values nor the means of other successive periods occur in random order. This lack of random sequence is explained below and a method of correcting for it is discussed.

EFFECT OF PERSISTENCE IN HYDROLOGIC DATA

Experience in our daily lives verifies the fact that rainy days occur together and dry days occur together. For similar meteorological reasons wet years tend to occur in groups and dry years similarly occur together. This tendency for grouping is called persistence. It is clear that if the means of groups that included a non-random assortment of individuals were computed, the spread or deviation among the means would be greater than if the groups consisted of a random selection of individuals. As an example, imagine measurements of the mean height of 10-men groups on a college campus Suppose the groups were made up by a random process so that each individual group of 10 consisted of some short men and some tall ones. In contrast, suppose that one of the 10-man groups consisted of the football team, another the basketball team, a third the coxswains of the crew, and so forth. The mean height of the basketball players would be larger than the mean of 10 randomly selected individuals. The mean of the coxswains would be smaller than the mean of a randomly selected group. Thus, the variation among the means of 10-men groups would be larger in the team groups—the nonrandomly selected groups—than in the randomly selected ones. Likewise, the spread of means of groups consisting of wet years and those consisting of dry years would be greater than if the groups consisted of randomly ordered individual years.

That the variability of groups of streamflows in their natural order of occurrence is actually larger than if the same flow values occurred in random sequence was sharply brought to the attention of the engineering profession by a distinguished British engineer, H. E. Hurst (1950). By working with the longest record of river stage in the world, the 1,050 years of recorded stage of the Nile at the Roda gage, Hurst obtained evidence that the tendency for wet years to occur together and dry years together increased variability of means of various periods. Other scientists confirmed this tendency with independent data.

Some records which show this effect are presented in table 3. These samples include some of the longest stream-discharge records in existence. Column 5 shows the standard deviation of the means of annual discharges for natural 5-year groups, for example, 1901–05, 1906–10, and so forth. Column 7 shows standard deviation of 10-year means in the same records, such as 1901–10, 1911–20, etc. Similarly, columns 9 and 11 are for 15- and 20-year groups.

Column 3 shows the standard deviation of annual discharge values. The standard deviation of annual values is unaffected by sequence; and, thus, the variability of annual values, expressed by the standard deviation, can be used as a yardstick against which the variability of means of 5-year, 10-year, and other periods may be compared. For this reason the ratio of variability of 5-year means to 1-year values is a factor that is independent of length of record, and various streams can be compared by their ratios as shown in column 6. Similarly, the ratio of standard deviation of 10-year means to that of annual values appears in columns 8, 10, and 12.

The data in table 3 were plotted to derive the dashed curve of figure 4. Ordinate values of individual points through which the dashed curve was drawn are not

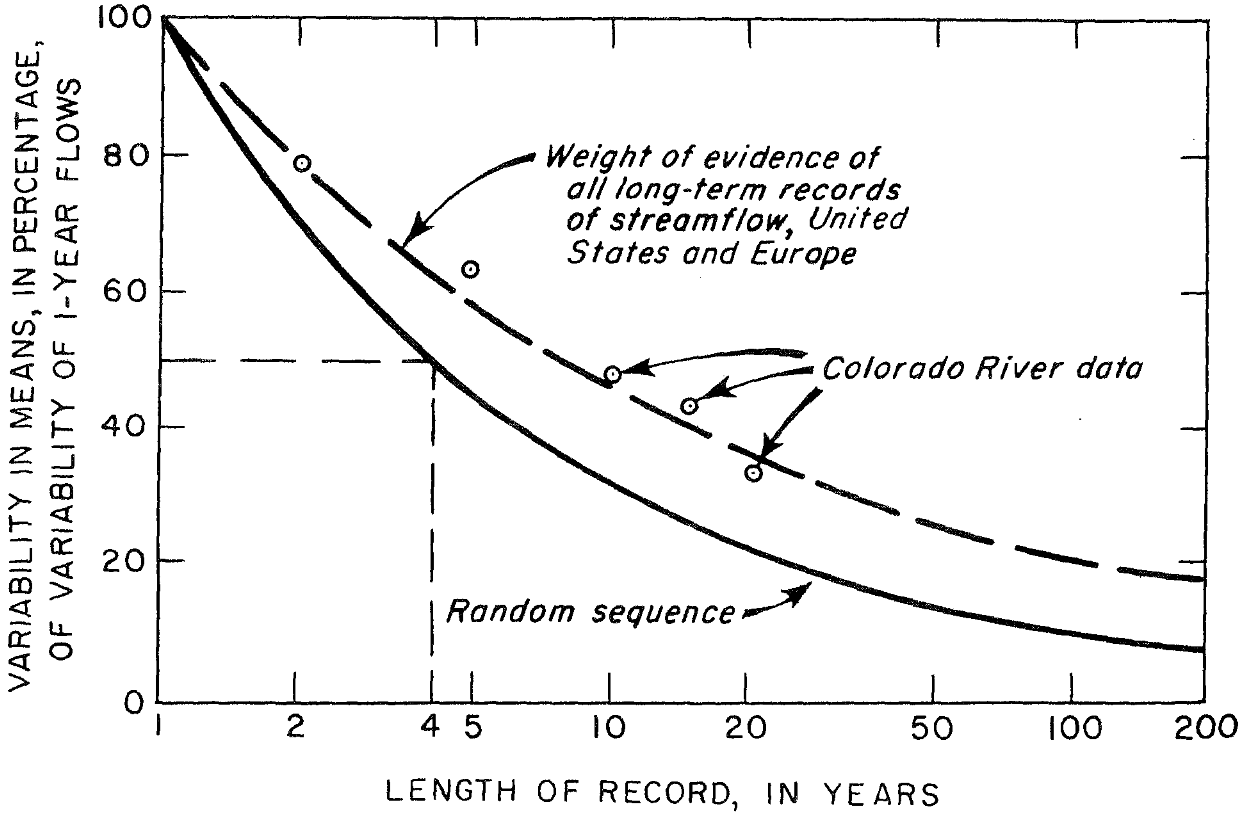


Figure 4.—Variability of mean values of streamflow for records of various lengths.

Table 3.—Variability of group means of streamflow data

Stream (1)	N (years) (2)	Standard deviation (3)	Coefficient of variation (4)	5-year means		10-year means		15-year means		20-year means		Units for standard deviation (13)	Source of data
				Standard deviation (5)	Ratio of standard deviation (6)	Standard deviation (7)	Ratio of standard deviation (8)	Standard deviation (9)	Ratio of standard deviation (10)	Standard deviation (11)	Ratio of standard deviation (12)		
Niagara River at Buffalo, N. Y.	93	16	0.078	118	0.74	10.6	0.66	8.9	0.56	9.1	0.57	1000 cfs	[1]
Tennessee River at Chattanooga, Tenn.	72	4.75	.20	2.44	.51	2.12	.45	1.79	.38	1.60	.34	inches	[2]
Columbia River near The Dalles, Oreg.	75	28	.20	18.7	.67	17.4	.62	15.6	.56	16.8	.60	million acre-ft	[2]
Vänern River at Sjötorp, Sweden.	144	100	.19	57	.57	39.3	.39	26.9	.27	25	.25	m^3 per sec	[3]
Rhine River at Strasburg, France.	70	210	.20	96	.46	76	.36	70	.33	60	.29	m^3 per sec	[4]
Connecticut River at Thompsonville, Conn.	104	3970	.208	2283	.58	1820	.46	1650	.42	1640	.41	cfs	[2]
Colorado River at Lees Ferry, Ariz.	61	4.2	.27	2.7	.64	2.0	.48	1.8	.43	1.4	.33	million acre-ft	[2]
Lake Cochituate Outlet at Cochituate, Mass.	87	7.4	.28	4.46	.60	3.15	.43	2.99	.40	2.55	.34	cfs	[2]
Spokane River at Spokane, Wash.	59	1960	.29	1194	.61	816	.42	613	.31	516	.26	cfs	[2]
Mississippi River at St. Louis, Mo.	95	38.2	.30	7.1	.45	13.9	.36	9.2	.24	8.4	.22	million acre-ft	[2]
Santa Ana River near Mentone, Calif.	58	40	.64	24.6	.61	25	.62	16	.40	....	....	cfs	[2]
Red River at Grand Forks, N. Dak.	70	1500	.64	1096	.73	1000	.67	805	.54	....	....	cfs	[2]
Verde River below Bartlett Dam, Ariz.	62	480	.66	237	.50	180	.38	189	.39	137	.28	cfs	[2]
Mode..........................	74	....	....	....	.58	....	.44	....	.40	....	.34	....	....

[1] U. S. Lake Survey.
[2] U. S. Geological Survey.
[3] Sveriges Meteorologiska och Hydrologiska Institut.
[4] Service de la Ponts et Chaussees.

shown; the ordinates are, for example, values shown in column 6, table 3, and a corresponding abscissa of 5 years. For Niagara River at Niagara Falls, for example, 5-year means (abscissa value of 5) have a ratio of 0.74 or 74 percent of the variability of the annual values in the same record. The dashed curve was extended beyond the 20-year abscissa to follow a smooth logarithmic line with slope of -0.35.

The actual values of the Colorado River at Lees Ferry (table 3) are plotted as circles in figure 4. For example, in 61 years of record, the 12 five-year means had a standard deviation, which was 64 percent of the standard deviation of the 61 annual values. Thus the cross at abscissa value of 5 has an ordinate position of 64 percent.

The circles representing Colorado River data plot consistently with the dashed line representing the data for the other rivers in table 2. The above results can be compared with similar results from a group of data that are not only normally or randomly distributed about their mean, but are randomly ordered as well. If group means are comprised of randomly chosen individuals, then the variability of these group means would decrease as the square root of the number of individuals making up the groups. Thus, if annual values of streamflow were randomly ordered—occurred in random sequence—variability of means of groups would decrease inversely as the square root of the number of years comprising the group. Thus, the means of 100-year groups would be $\frac{1}{\sqrt{100}}$ or $\frac{1}{10}$ as variable as 1-year values.

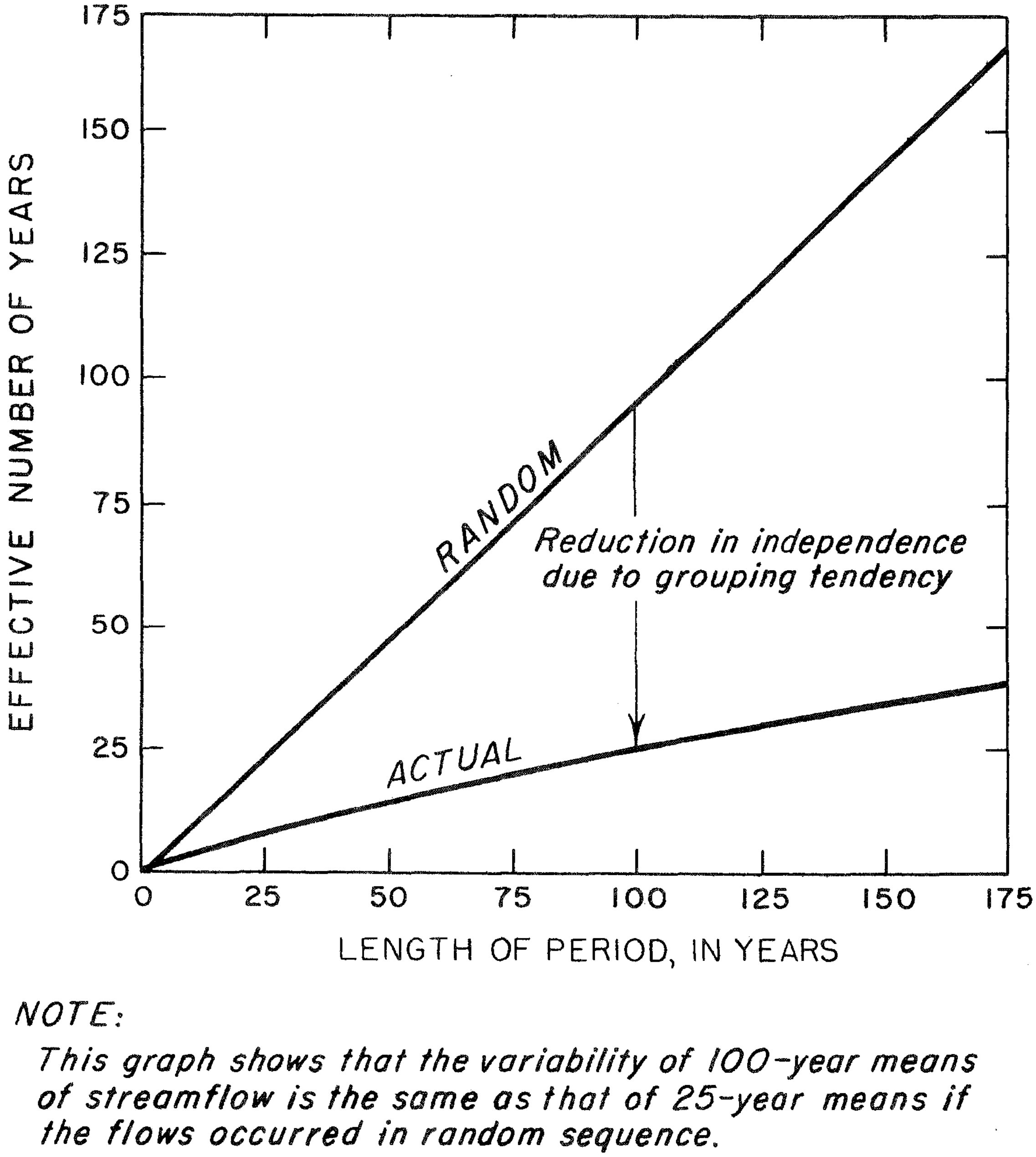


Figure 5.—The effect of grouping tendency in streamflow data.

Randomly ordered data are described by the solid line in figure 4. Relative to the variability of 1-year values, means of 100-year records would be 1/10 as variable. Thus, the solid line goes through the ordinate value of 10 percent for an abscissa value of 100 years.

The difference between the variability of naturally occurring groups of streamflow data and the same data randomly ordered is defined by the difference between the dashed line and solid line of figure 4. For periods of equal length (for example, 5-year group averages) the variability (ordinate value) is larger in naturally ordered data than the same values randomly ordered.

The same idea can be presented in another way, as shown on figure 5. An actual record of 100 years has about the same variability as means randomly ordered in a 25-year record. Similarly, the means of 200 years of actual records have about the same variability as the means of 40 years if these discharges occurred in random order rather than in their natural order. A 100-year record is, therefore, required to estimate the mean flow with the same confidence as could be estimated from 25 years of record were the flows to occur in random order. After a record exceeds 100 years in length (fig. 5), the estimate of the long-term mean improves at a decreasing rate—that is, the effectiveness of a record of 200 years is but 20 percent better than that of an estimate based on 100 years of record.

To summarize, the tendency toward persistence in meteorologic and climatic events leads to a larger variability of mean values than if the same events occurred in random order. A group of values, such as annual discharge quantities, may be normally distributed about their mean value but occur in nonrandom sequence. The greater variability caused by this linkage is similar in data from rivers of widely different size and type; and as shown below, the similarity between rivers provides a means of correcting for this linkage in making probability analyses of any given streamflow record.

PROBABLE VARIATION AMONG MEANS OF FUTURE SAMPLES

Variability is defined as the dispersion or spread of values about their mean. Thus, in figure 3 the slope of the line on the graph is a measure of the variability in the sample data. Figure 4 defines the variability of means of groups of various sizes (periods of years) in terms of the variability of annual streamflow quantities. This ratio, read as percentage on the ordinate of figure 4, can be used to express the slope of a line representing the probable dispersion of means of future samples. The percentage is merely multiplied by the slope of the line showing distribution of annual values to obtain the slope of a line representing the dispersion of means of 5-year, 10-year, or some other period.

The dispersion of 10-year means and 61-year means is indicated by appropriate lines on figure 6 as examples. For comparison, the dispersion of the annual values is also shown. The position and slope of the line representing annual values is the same as that shown in figure 3. It will be recalled that the slope of that line can be expressed by two points. One point is the mean plotted at 50-percent probability. The second point is the mean, 15.18 million acre-feet, plus the probable deviation, 2.84 million acre-feet, plotted at 75-percent probability.

To obtain the slope of the line representing 10-year means, the 10-year abscissa in figure 4 is read from the dashed curve representing actually ordered or linked hydrologic data; a value of 44 percent is indicated. The slope would be determined as follows:

$$0.44 \times 2.84 = 1.25 \text{ million acre-feet (probable deviation)}$$

Thus, if the mean for the 61 years is the true mean, 75 percent of 10-year means would be expected to be equal to, or less than,

$$15.18 + 1.25 = 16.43 \text{ million acre-feet.}$$

In figure 6, the 10-year line has an ordinate value of 16.43 at an abscissa value of 75 percent.

Similarly, the line representing the dispersion of means of 61-year periods will have an ordinate value of

$$15.18 + (0.24 \times 2.84) = 15.86$$

corresponding to an abscissa value of 75 percent. This can be interpreted as follows: Assuming the 61-year mean to be the true mean, three-fourths of the discharge values representing means of 61-year periods will be equal to or less than 15.86 million acre-feet. By the same token, one-fourth of the means of future 61-year periods will be equal to or less than

$$15.18 - (0.24 \times 2.84) \approx 14.5 \text{ million acre-feet.}$$

Throughout this discussion it will be understood that the computations are for the reconstructed record of virgin flow at Lees Ferry and the actual runoff will be less than these figures owing to upstream depletion.

CONFIDENCE IN ESTIMATE OF FUTURE VARIABILITY

In figure 6 shows the most probable distribution of means of future 10-year and 61-year periods, assuming the 61-year mean to be the true mean. The phrase "most probable" implies that the true variation, which actually will be experienced, may be somewhat different. This is reasonable because an estimate made from a sample would be unusual if it were a perfect expression of the whole population. Probability theory allows an objective estimate of the expected deviation of any sample or group of samples from the true attributes of the whole population. In the previous analysis, the sample mean was assumed to be the same as the true mean. The probability that the sample mean may differ from the true mean is considered below.

The objective estimate of sampling differences is called the confidence limit or confidence band. Confidence limits are derived from the characteristic of normal distributions already employed, so that the standard error of the mean of a sample is an approximation of the standard deviation of the means of many samples of equal size.

The mean value of the 61-year sample of annual discharge values is 15.18 million acre-feet. The probable error (p.e.) of this mean is equal to

$$0.24 \times 2.84 = 0.68 \text{ million acre-feet.}$$

Stated more simply, there is a 50 percent chance that the true mean of the whole population (indefinitely long

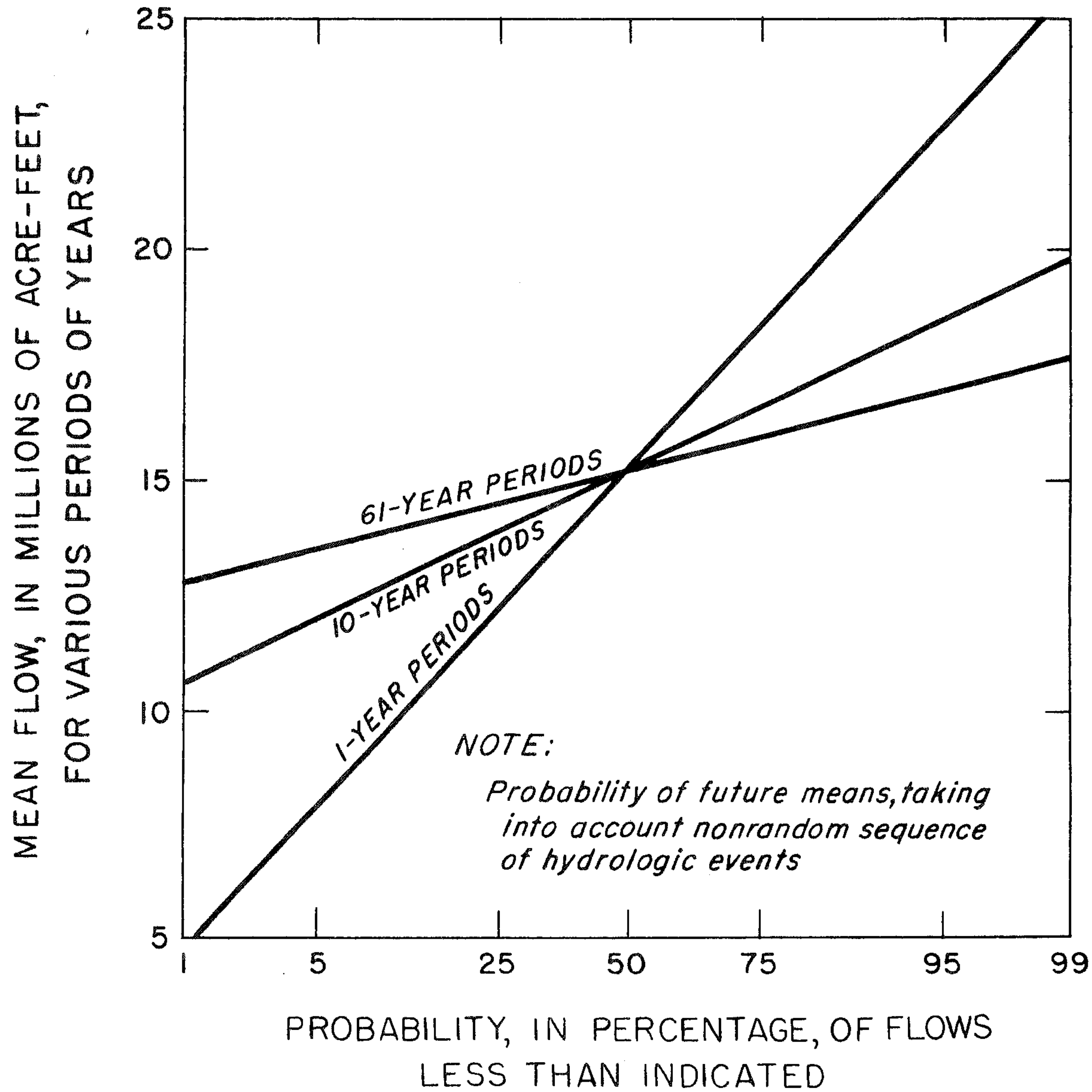


Figure 6.—Probable distribution of mean discharge values for periods of various lengths, Colorado River at Lees Ferry.

period of time) lies within one probable deviation on either side of the mean of the sample, or

$15.18 \pm 0.68 = 14.50-15.86$ million acre-feet.

Thus, the line representing the most probable distribution of means of future 61-year periods drawn on figure 6 could lie in a slightly upward or downward position on the chart. There is a 50 percent chance that its true position is within a spread upward or downward of the mean by an amount equal to 0.24 probable deviation or 0.68 million acre-feet.

As defined earlier, there is an 82-percent chance that a limit of twice this value above or below the sample mean would include the true mean of the whole population. In such a manner bands expressing any desired confidence could be defined. The present discussion deals with only one of these various possible confidence limits that represents a 50-percent chance.

The 50-percent confidence band has been drawn on figure 7 as a pair of parallel lines lying 0.68 million acre-feet above and below the mean at the abscissa value of 50 percent. The slope of the parallel lines was determined previously; that is, the ordinate value at 75-percent probability is 0.68 million acre-feet higher than the ordinate at 50-percent probability. Thus, the confidence-limit lines are parallel to the line on figure 6 that represented the most probable dispersion of means of future 61-year periods.

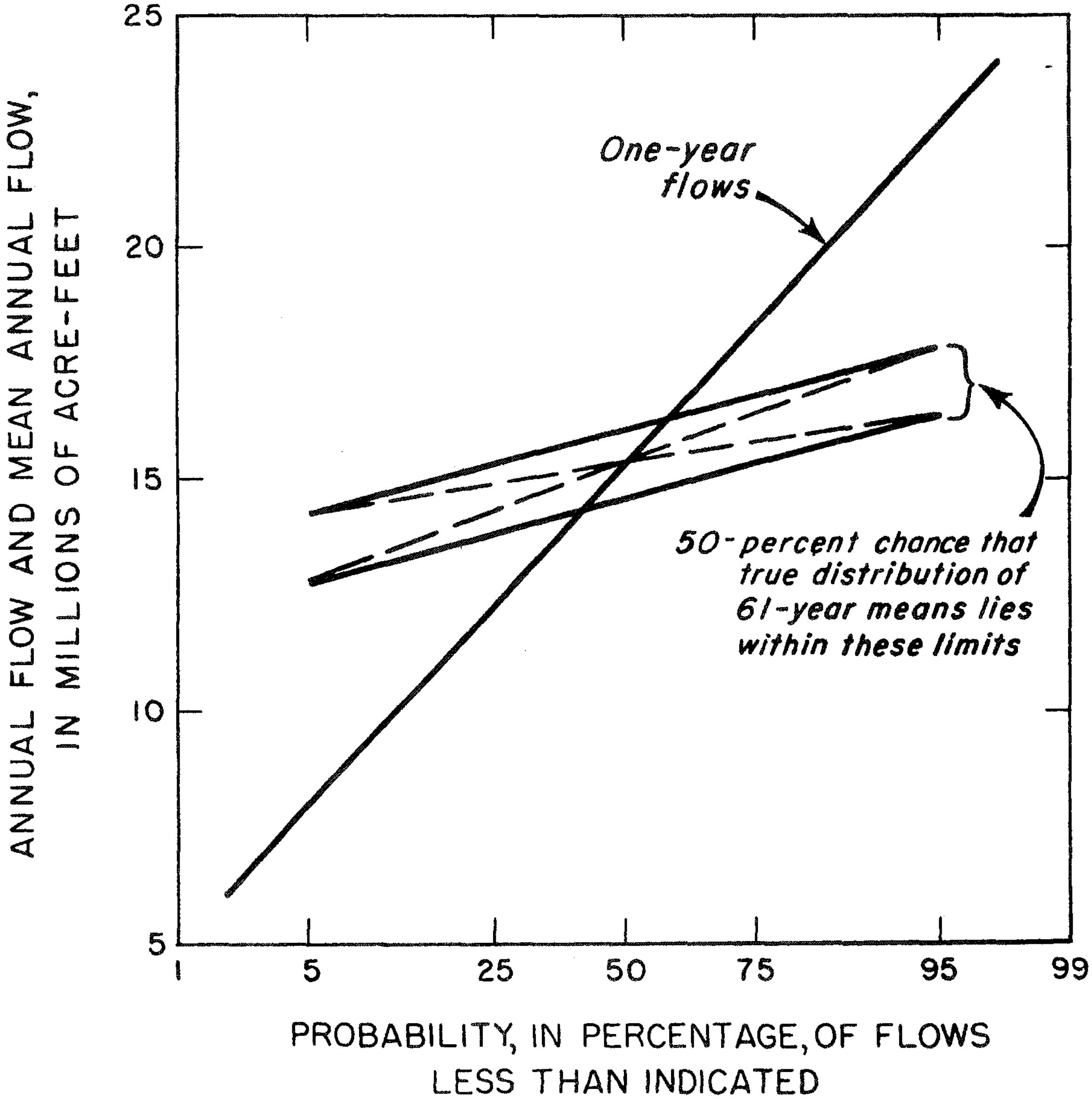


Figure 7.—Variability of 61-year means, Colorado River at Lees Ferry.

A more elegant but only slightly different construction of confidence limits yields curved lines rather than the parallel straight lines of figure 7.

Because a sample yields only an estimate of value of the true mean of the whole population and only an estimate of the dispersion of the means of other samples drawn from the same population, there is a 50-percent chance that the line representing the actual distribution of means of future samples may be anywhere within the band defined by the confidence limits. Two possible positions of the line representing such a distribution are shown by the dashed lines within the confidence band shown on figure 7. An infinite number of such possible positions exist, having various slopes within the limit of the confidence band and having various vertical positions within that band. Any of these possible positions are equally probable; therefore, according to the assumptions used in figure 7, there is also a 50-percent chance that the distribution of future 61-year means will lie outside of the confidence band drawn.

PROBABLE VALUE OF MEAN FLOW IN NEXT 61-YEAR PERIOD

When the probable value of the mean of some prospective period in the future, such as the next time period, is discussed, then it must be considered that the period will also be a sample with the same probability of variation from the true mean as was the sample already available. The variation from the true value of the mean of the available sample must be coupled with the variation in the next sample to obtain the total possible variation from the true, but unknown, value applicable to the whole population.

These errors—used in a statistical sense—are not added together but combined as the square root of the sums of their respective squares.

To present a practical example, in a water-supply problem the engineer is interested in the probability of the mean discharge of the next period of time being higher or lower than the mean value during the period of available record. He is particularly interested in estimating the probability of a lower value of streamflow in the next period than he observed in the last one, because in water-supply problems, deficient flow can be critical. In the case of the 61-year period of the Colorado River at Lees Ferry, compute the lowest value that the mean of the next 61 years is likely to be in 1 chance out of 4. Inasmuch as the variability of means of particular future 61-year periods relative to the variability of annual data from a 61-year sample equals the statistical sum of the variability of the sample and the future periods,

$$\sqrt{(0.24)^2 + (0.24)^2} = 0.34.$$

The width of the 50-percent-confidence band is, thus,

$$0.34 \times 2.84 = 0.96 \text{ million acre-feet}$$

above and below the mean already experienced in the available sample. Therefore, there is a 50-percent chance that the next 61-year mean lies between 15.18 ± 0.96 million acre-feet or between 16.14 and 14.22 million acre-feet. Thus, it can be said that there is a 25-percent chance, or 1 chance in 4, that the mean of the next 61 years would be less than 14.22 million acre-feet.

By a similar procedure the variability means of future 10-year periods can be computed and the lower limit for the probable mean value of the next 10-year period can be found. In this case, the combined variability would be

$$\sqrt{(0.24)^2 + (0.44)^2} = 0.50.$$

There is a 50-percent chance that the mean flow during a specific 10-year period, such as the next 10-year period, will be

$$15.18 \pm (0.5 \times 2.84) = 16.60\text{–}13.76 \text{ million acre-feet.}$$

There is also an 82-percent chance that the next 10-year mean will lie within two probable deviations from the sample mean or

$$15.18 \pm (2 \times 0.5 \times 2.84) = 18.02\text{–}12.34 \text{ million acre-feet.}$$

Therefore, it can be stated that there is a 9-percent chance (1 out of 11) that the next 10-year mean will be less than 12.34 million acre-feet.

With regard to the record of the Colorado River at Lees Ferry, an inquiry could be made as to whether the means actually experienced during the driest 10-year period of record greatly exceeded reasonable expectations. The lowest 10-year period was the decade 1931-40, with a mean of 11.83 million acre-feet. This discharge was only slightly below that for the 9-percent probability. Thus, it may be concluded that the lowest 10-year mean in the 61-year period at Lees Ferry might have been expected in 1 chance out of 11—a reasonable probability of occurrence.

EFFECT OF STORAGE ON STREAMFLOW VARIABILITY

Variability of discharge is an inherent characteristic of rivers. Storage reservoirs are devised by man to make variable river flows match his needs for water; that is, water-supply reservoirs are built to hold over water from wet periods in order that it may be discharged during dry ones. Reservoir storage, therefore, is merely a feasible way for man to reduce the natural variability of stream discharge.

In figure 8, the line having the greatest slope is the most probable distribution of 10-year means and is identical with the 10-year line of figure 6; it may be considered to represent what nature has provided. The line having the lesser slope (fig. 8) is arbitrarily drawn on the graph to indicate the lower variability, which man desires to achieve by reservoir storage. The hatched area between the two lines is a quantitative measure of the regulation achieved by storage. Absolutely even flow would be represented in figure 8 by the horizontal dashed line, but it is theoretically and physically impossible to achieve a uniform outflow because, as indicated on figure 4, even long periods, say 200 years, have means which are likely to vary by a considerable amount. Thus, if successive 200-year periods have considerably different mean values of flow, a relatively large amount of reservoir storage would have to be built to hold over water, which occurred as high flows in one 200-year period, to supply low periods in a succeeding 200-year period of relative dryness.

By the same reasoning, each additional increment of storage capacity yields a smaller and smaller increment of actual flow regulation; that this is the actual experience with reservoirs built and operated in the United States is shown in figure 9 (Langbein, 1959). The ordinate of this chart represents the present regulation, in which regulation is defined as the average of the total increments to storage that occurred on an annual basis. Regulation on this graph is expressed as a percentage of the mean annual flow. It would be physically impossible to add, on the average, more water to storage than the average annual flow of the stream. For this reason, ordinate values cannot possibly exceed 100 percent of the mean annual flow.

The abscissa scale represents the storage capacity, expressing capacity as a ratio to mean annual flow. Each point represents reservoir data tabulated in table 4. The reservoirs listed in table 4 were chosen to represent a reasonable sample of different kinds of reservoirs and have a variety of storage capacities in terms of the annual flow of the respective streams. Reservoirs built exclusively for flood control are not included. The point representing Lake Mead, for example, is plotted at an abscissa value of about 2 inasmuch as the 29 million acre-feet of usable storage in that reservoir represents approximately 2 times the annual mean discharge of the river. The ordinate value for Lake Mead represents the average annual increment to storage actually experienced since the reservoir was built. It will be noted that the operating experience at Lake Mead, represented by a point on figure 9, fits reasonably well with the points for other reservoirs plotted on this chart.

Table 4.—Capacity and regulation of some representative reservoirs

Reservoir and State	Usable capacity		Mean annual regulation[1]		
	Acre-feet[2]	Detention period (years)[3]	Acre-feet per year	Ratio to capacity	Ratio to mean annual flow
Piney, Pa	13,000	0.011	70,000	5.4	0.06
Great Falls, Tenn	49,400	.021	204,000	4.1	.089
Ocoee No. 1, Tenn	33,000	.035	204,000	3.6	.13
Claytor, Va	100,000	.04	150,000	1.5	.06
Mascoma Lake, N. H	7,744	.05	22,000	2.8	.14
Franklin D. Roosevelt Lake, Wash	5,072,000	.07	4,800,000	.95	.07
West Fork Bitterroot, Mont	31,700	.14	26,000	.82	.12
Hiwassee, N. C	1,376,000	.265	330,000	.90	.24
Green Mountain, Colo	146,900	.34	111,000	.75	.26
Gibraltar, Calif	7,731	.39	4,300	.55	.21
Stillwater, N. Y	106,000	.40	91,000	.86	.34
Sacandaga, N. Y	762,300	.51	560,000	.75	.38
First and Second Connecticut Lakes, N. H.	88,106	.60	62,000	.70	.42
Norris, Tenn	2,281,000	.72	1,070,000	.47	.34
Shasta Lake, Calif	4,377,000	.80	1,530,000	.35	.28
Lake Alamanor, Calif	649,800	1.0	250,000	.38	.38
Salmon River Canal Co., Idaho	182,650	1.6	57,500	.31	.54
Henrys Lake, Idaho	79,351	2.1	20,600	.26	.54
Lake Mead, Ariz.–Nev	27,207,000	2.1	[4]5,750,000	.21	.44
Lake Mead plus Lake Mohave	29,000,000	2.3	[4]6,500,000	.22	.51
Fort Phantom Hill, Tex	69,500	2.3	11,900	.175	.40
Lake Kickapoo, Tex	106,000	2.6	20,200	.19	.49
Elephant Butte, N. Mex	2,185,000	2.6	[4]375,000	.17	.45
Elephant Butte plus Caballo, N. Mex	2,526,000	3.0	[4]470,000	.185	.55
Quabbin, Mass	1,279,000	6.0	119,000	.09	.56
San Carlos, Ariz	1,205,000	6.7	[4]117,000	.097	.65
Lake Henshaw, Calif	194,320	21.8	8,170	.042	.91

[1]For reservoirs with monthly detention period greater than 0.1 year, regulation was computed from monthly changes in reservoir contents. Daily data were used for smaller reservoirs.

[2]Thomas and Harbeck, 1956.

[3]Ratio of usable capacity to mean annual flow.

[4]Including evaporation losses.

The smooth line representing the general experience of reservoir operation has been drawn through the points; it approaches but does not reach the ordinate value of 100 percent, as has been explained above. Thus, both theoretically and from actual operating experience, complete regulation yielding the mean annual flow of the stream is impossible to achieve. The smooth line in figure 9 confirms that successive increments of reservoir capacity add increasingly smaller increments to regulation.

In plotting figure 9, evaporation losses from the reservoirs have been computed and included in the annual regulation. If the regulation less evaporation losses were computed, the smooth curve drawn in figure 9 would lie at a lower value and become asymptotic or even fall away from the horizontal line where large values of reservoir capacity are shown.

By applying the generalized experience indicated by the smooth curve in figure 9 to the Colorado River, ordinate values have been read off the smooth curve and used to compute the increments of regulation that would be attained by different assumed reservoir capacities constructed in the upper Colorado River basin. The results of this computation, again including evaporation losses as a part of the annual regulation, define the solid line in figure 10. Evaporation losses subtracted from ordinate values yield net regulation as defined by the dashed line. It can be seen from this dashed line that total reservoir capacity in excess of about 40,000,000 acre-feet would achieve practically no additional water regulation if evaporation loss is subtracted from annual regulation. Thus, generalized experience with representative reservoirs in the United States indicates that if reservoirs with capacity beyond an additional 10 million to 15 million acre-feet are constructed in the upper Colorado River basin, evaporation loss will thereafter offset the hydrologic benefit of the regulation so achieved.

REFERENCES

Hurst, H. E., 1950, Long-term storage capacity of reservoirs: Am. Soc. Civil Engrs. Proc., v. 76, Separate 11.

Langbein, W. B., 1959, Water yield and reservoir storage in the United States: U. S. Geol. Survey Circ. 409.

Thomas, N. O., and Harbeck, G. E. Jr., 1956, Reservoirs in the United States: U. S. Geol. Survey Water-Supply Paper 1360-A.

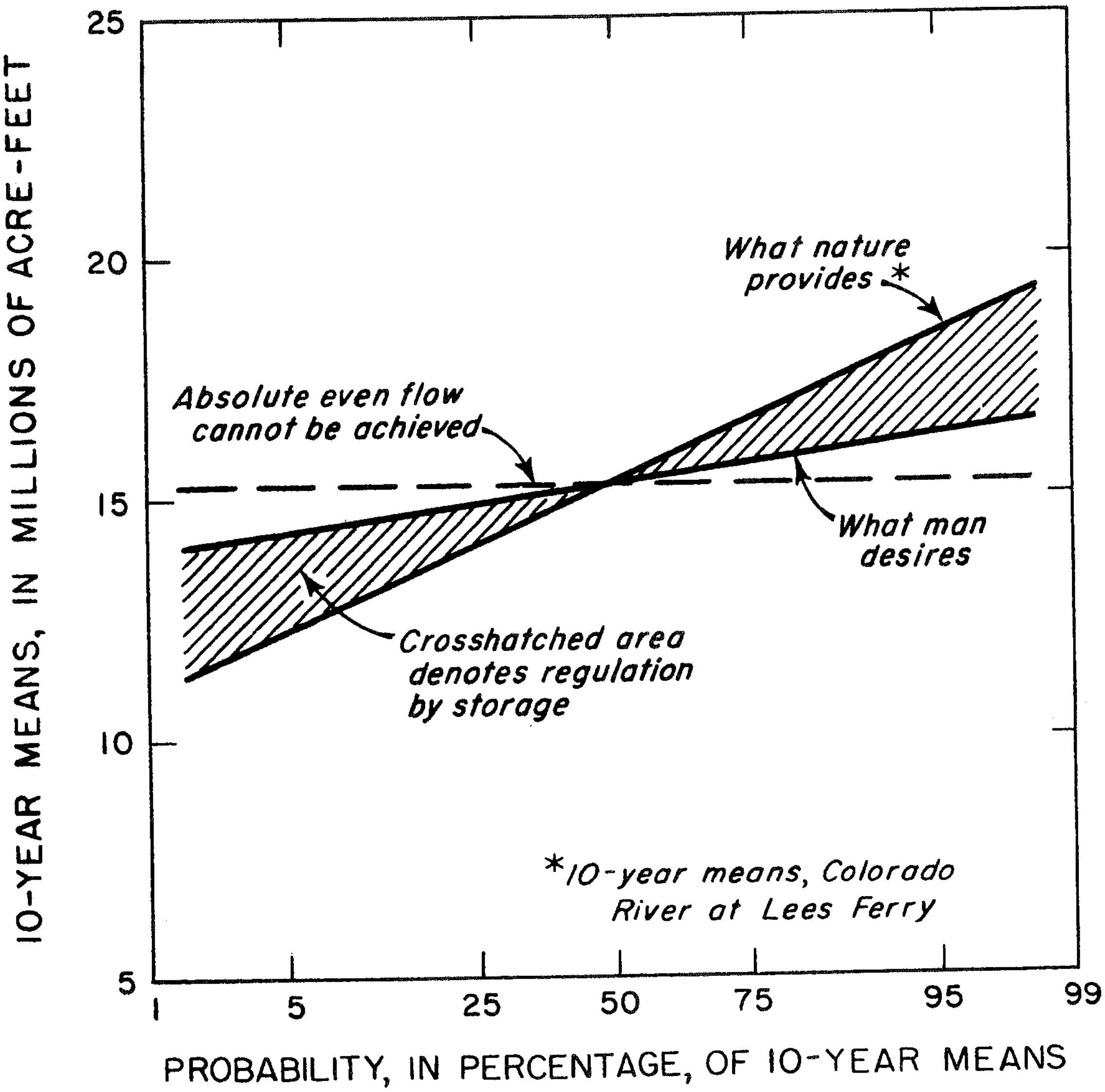


Figure 8.—The decrease of variability by storage.

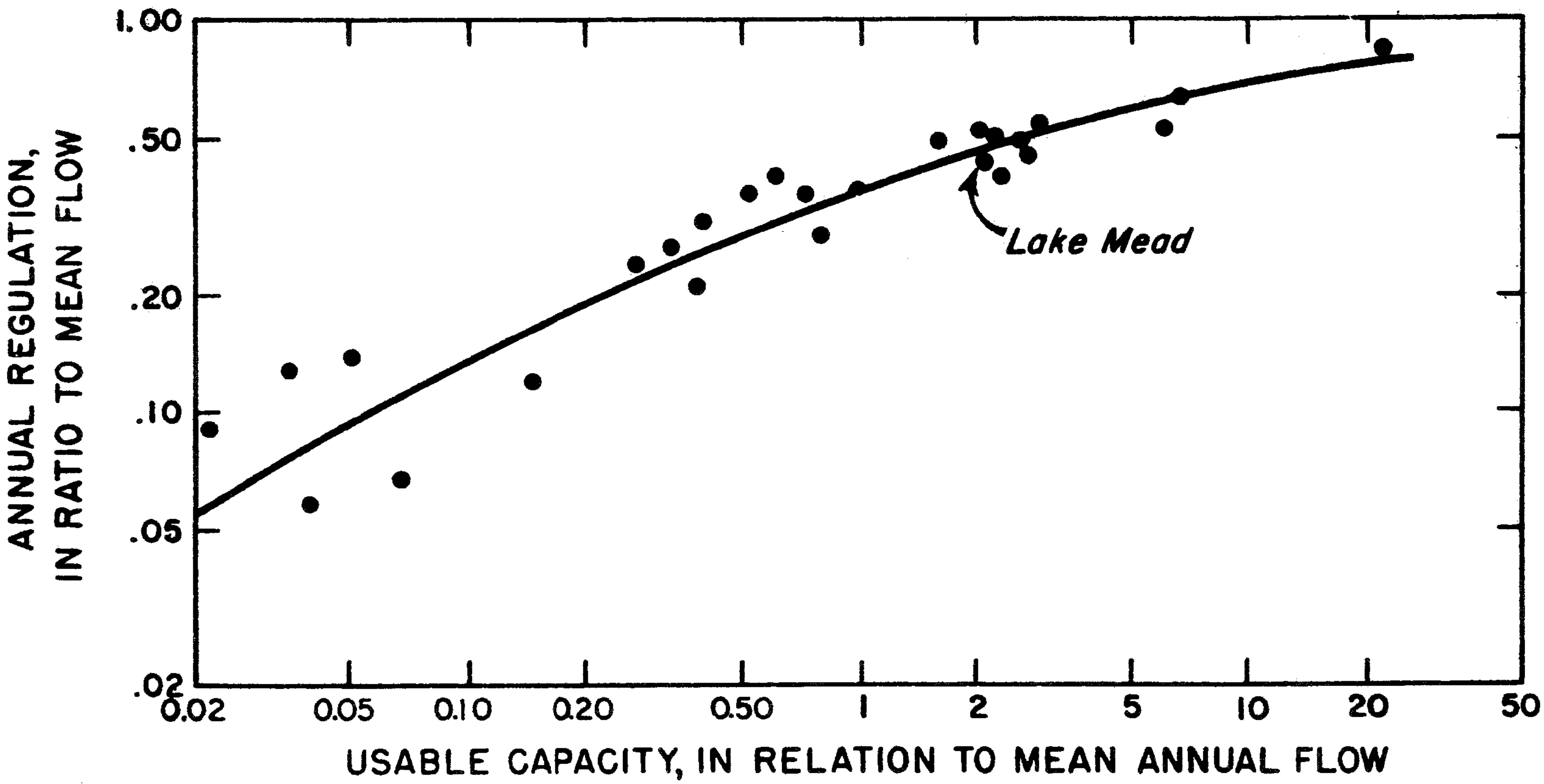
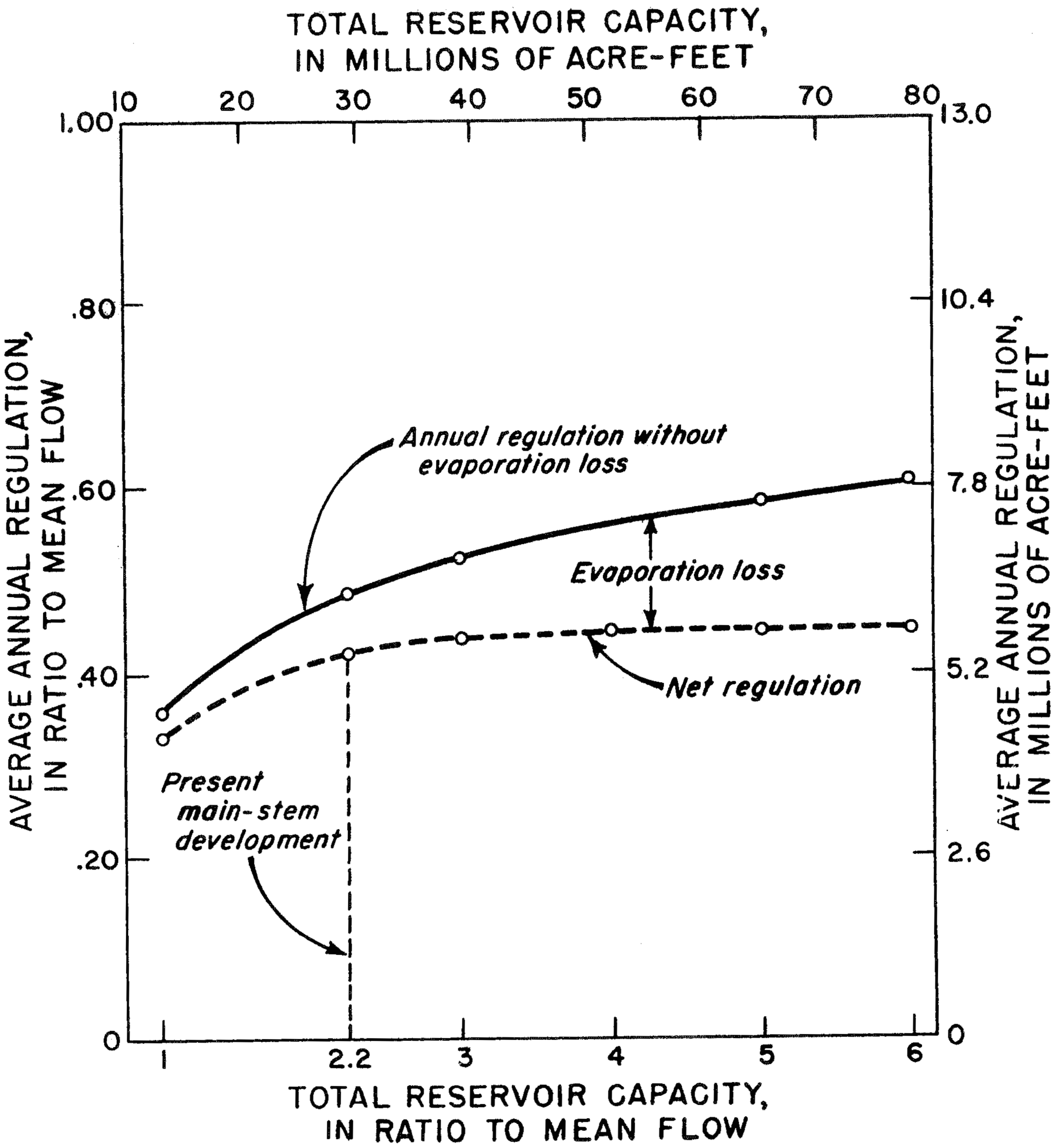


Figure 9.—The relation of usable capacity to annual regulation, representative reservoirs in the United States (Langbein, 1959).



NOTE: For computation, historic mean flow taken as 13 million acre-feet, representing depleted flow at Lees Ferry; evaporation loss assumed 27,000 acre-feet per year per million acre-feet of reservoir capacity.

Figure 10.—The effect of various amounts of storage capacity on flow regulation, Colorado River basin.

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